

Coordinatization structures for generalized quadrangles and glued near hexagons

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Abstract

A generalized admissible triple is a triple $T = (\mathcal{L}, X, \Delta)$, where X is a set of size $s + 1 \geq 2$, \mathcal{L} is a Steiner system $S(2, s + 1, st + 1)$, $t \geq 1$, with point-set P and Δ is a very nice map from $P \times P$ to the group $Sym(X)$ of all permutations of the set X . Generalized admissible triples can be used to coordinatize generalized quadrangles with a regular spread and glued near hexagons. The idea of coordinatizing these incidence structures in this way has led to several breakthrough results in the theory of generalized quadrangles and near polygons. Generalized admissible triples allow to give unified constructions for several classes of generalized quadrangles, they allow to classify all generalized quadrangles of order 5 with a center of symmetry and they also allow to characterize the symplectic generalized quadrangle $W(q)$ as this generalized quadrangle of order q having a hyperbolic line consisting of only regular points. In this chapter, we give a survey of the theory of generalized admissible triples and of many of its interesting applications.

1 Introduction

1.1 Basic definitions

A *(point-line) incidence structure* is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ with \mathcal{P} a nonempty set whose elements are called *points*, \mathcal{L} a possibly empty set whose elements are called *lines* and I a subset of $\mathcal{P} \times \mathcal{L}$ called the *incidence relation*. If

$(p, L) \in I$, then we say that p is incident with L , that p is contained in L , that L contains p , etc. The *point-line dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ with nonempty line-set \mathcal{L} is the incidence structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, I^D)$, where $\mathcal{P}^D = \mathcal{L}$, $\mathcal{L}^D = \mathcal{P}$ and $I^D = \{(L, p), \mid (p, L) \in I\}$.

A point-line incidence structure is called a *partial linear space* (respectively a *linear space*) if every two distinct points are incident with at most (respectively exactly) one line. A partial linear space is said to be of *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines. A *Steiner system* $S(t, k, v)$ is an incidence structure with v points satisfying the property that every t distinct points are incident with precisely one line and every line is incident with precisely k points.

The *point graph* or *collinearity graph* of a point-line incidence structure \mathcal{S} is the graph whose vertices are the points of \mathcal{S} with two different points adjacent whenever they are collinear, i.e. whenever there exists a line incident with these points.

Let $\Gamma = (V, E)$ be a simple undirected graph without loops. A *clique* of Γ is a set of mutually adjacent vertices. A clique is called *maximal* if it is not properly contained in another clique. We will denote the distance between two vertices x and y of Γ by $d(x, y)$. If X_1 and X_2 are two nonempty sets of vertices, then we denote by $d(X_1, X_2)$ the minimal distance between a vertex of X_1 and a vertex of X_2 . If X_1 is a singleton $\{x_1\}$, then we will also write $d(x_1, X_2)$ instead of $d(\{x_1\}, X_2)$. For every $i \in \mathbb{N}$ and every nonempty set X of vertices, we denote by $\Gamma_i(X)$ the set of all vertices y for which $d(y, X) = i$. If X is a singleton $\{x\}$, then we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A *near $2d$ -gon* is a connected graph of finite diameter d with the property that for every vertex x and every maximal clique M there exists a unique vertex x' in M nearest to x . A near 0-gon consists of one vertex and a near 2-gon is just a complete graph with at least two vertices.

There is a bijective correspondence between the class of near polygons and a class of partial linear spaces. If a graph Γ is a near polygon, then the point-line incidence structure with points, respectively lines, the vertices, respectively maximal cliques of Γ (natural incidence) is a partial linear space \mathcal{S} . The graph Γ can easily be retrieved from \mathcal{S} : Γ is the point graph of \mathcal{S} . Because of this bijective correspondence, we will call the partial linear spaces which correspond to near polygons also near polygons. In this chapter, we will always adopt the geometric point of view. A near 0-gon is a point and a near 2-gon is a line. We will denote the unique line with $s + 1$ points by

\mathbb{L}_{s+1} . Near quadrangles are usually called *generalized quadrangles*. If x is a point and if L is a line of a near polygon, then we denote by $\pi_L(x)$ the unique point of L nearest to x .

If $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ are two near polygons with $\mathcal{P}_1 \cap \mathcal{L}_1 = \emptyset = \mathcal{P}_2 \cap \mathcal{L}_2$, then we can define the following incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$:

- $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$;
- $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$;
- the point (x, y) of \mathcal{S} is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $(y, L) \in \mathcal{I}_2$, the point (x, y) is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $(x, M) \in \mathcal{I}_1$ and $y = u$.

We denote \mathcal{S} also by $\mathcal{S}_1 \times \mathcal{S}_2$ and call it the *direct product* of \mathcal{S}_1 and \mathcal{S}_2 . An $(n_1 \times n_2)$ -grid is the direct product of a line of size n_1 and a line of size n_2 . An $(n_1 \times n_2)$ -grid is called *symmetrical*, respectively *nonsymmetrical*, if $n_1 = n_2$, respectively $n_1 \neq n_2$. The point-line dual of a (symmetric, nonsymmetrical) grid is called a (*symmetrical*, *nonsymmetrical*) *dual grid*.

For more background information on general near polygons, we refer to the recent book [4] of the author. For more background information on generalized quadrangles, we refer to Payne and Thas [13].

1.2 Generalized quadrangles

As mentioned above, generalized quadrangles are near quadrangles. They are point-line incidence structures which satisfy the following properties:

- (GQ1) Every two distinct points are incident with at most 1 line.
- (GQ2) There exists a non-incident point-line pair.
- (GQ3) For every non-incident point-line pair (p, L) , there exists a unique point $\pi_L(x)$ on L collinear with x .

As generalized quadrangles will play a key role in this chapter, it is necessary to discuss these incidence structures in greater detail.

A generalized quadrangle is called *degenerate* if there exists a point x which is collinear with all the remaining points. In this case, x is incident with every line of the generalized quadrangle.

Almost all generalized quadrangles have an order:

Proposition 1.1 ([4, Theorem 1.14]) *One of the following cases occurs for a generalized quadrangle Q :*

- Q is degenerate;
- Q is a nonsymmetrical grid;
- Q is a nonsymmetrical dual grid;
- Q has an order (s, t) .

A generalized quadrangle of order (s, t) will shortly be denoted as $\text{GQ}(s, t)$. A generalized quadrangle of order (s, s) is also called a generalized quadrangle of order s .

A *triad* of a generalized quadrangle Q is a set $T = \{x, y, z\}$ of three mutually noncollinear points. Any point of Q collinear with x , y and z is called a *center* of T .

Let Q be a generalized quadrangle of order (s, t) . For every point x of Q , let x^\perp denote the set of all points collinear with x (so $x \in x^\perp$). If X is a nonempty set of vertices of Q , then we define $X^\perp := \bigcap_{x \in X} x^\perp$ and $X^{\perp\perp} := (X^\perp)^\perp$ if $X^\perp \neq \emptyset$. If x and y are two distinct points, then $|\{x, y\}^\perp|$ is equal to either $s + 1$ or $t + 1$ depending on whether x and y are collinear or not. The set $\{x, y\}^{\perp\perp}$ is called the *span* of the pair (x, y) . If x and y are collinear, then $\{x, y\}^{\perp\perp}$ coincides with the line xy . If x and y are not collinear, then $\{x, y\}^{\perp\perp}$ is also called the *hyperbolic line* through x and y ; since $\{x, y\}^\perp$ contains two noncollinear points, this hyperbolic line contains at most $t + 1$ points. If the hyperbolic line through two noncollinear points x and y contains precisely $t + 1$ points, then the pair (x, y) is called *regular*. A point x is called *regular* if the pair (x, y) is regular for every point y not collinear with x . If x is a regular point of a generalized quadrangle of order $s \geq 2$, then the incidence structure π_x with points the elements of x^\perp and with lines all the spans $\{a, b\}^{\perp\perp}$, $a, b \in x^\perp$ with $a \neq b$, is a projective plane of order s , see Payne and Thas [13, 1.3.1]. The point-line dual of a $\text{GQ}(s, t)$ is a $\text{GQ}(t, s)$; so, the notion of regularity can also be defined for the lines of a generalized quadrangle.

An *ovoid* of a generalized quadrangle Q is a set of points intersecting each line of Q in a unique point. The dual notion is that of spread. A *spread* S of a generalized quadrangle Q is a set of lines partitioning the point set of Q . A spread S is called *regular* if the following holds for any two different lines

K and L of S : (i) (K, L) is regular, (ii) $\{K, L\}^{\perp\perp} \subseteq S$. A spread S of Q is called a *spread of symmetry* if for every line $K \in S$ and all $k_1, k_2 \in K$, there exists an automorphism of Q fixing each line of S and mapping k_1 to k_2 . Every spread of symmetry is a regular spread. If S is a spread of a $\text{GQ}(s, t)$ with $t \neq 1$, then by De Bruyn [1, Theorem 4.1], there exist at most $s + 1$ automorphisms of the generalized quadrangle which fix each line of S , and equality holds if and only if S is a *spread of symmetry*.

If x is a point of a $\text{GQ}(s, t)$ with $s \neq 1$, then there are at most t automorphisms of the generalized quadrangle which fix every point of x^\perp , see Payne and Thas [13]. If there are precisely t such automorphisms, then x is called a *center of symmetry*. A center of symmetry is always a regular point.

Suppose ζ is a symplectic polarity of the 3-dimensional projective space $\text{PG}(3, q)$. The points and totally isotropic lines (with respect to ζ) of $\text{PG}(3, q)$ define a generalized quadrangle $W(q)$, which is called the *symplectic generalized quadrangle of order q* . Every point of $W(q)$ is a center of symmetry (and hence also a regular point).

If x is a regular point of a generalized quadrangle Q of order s with $s \neq 1$, then a new generalized quadrangle $P(Q, x)$ can be derived from it, see Payne [11] or Payne and Thas [13, 3.1.4]. The points of $P(Q, x)$ are the points of Q not collinear with x and the lines of $P(Q, x)$ are on the one hand the lines of Q not containing x and on the other hand the hyperbolic lines of Q through x (natural incidence). $P(Q, x)$ is a generalized quadrangle of order $(s - 1, s + 1)$ which is called the *expansion* of Q about x . The hyperbolic lines through x define a regular spread $S(Q, x)$ of $P(Q, x)$. By Theorems 2.7 and 2.8 of De Soete and Thas [9], the following are equivalent: (i) x is a center of symmetry, (ii) $S(Q, x)$ is a spread of symmetry of $P(Q, x)$.

Let Q be a generalized quadrangle of order $s \geq 2$ with a regular point x and let y be a point of Q noncollinear with x . Let u, v and w denote three points of Q noncollinear with x such that $y \sim u \sim v \sim w \sim y$, $y \not\sim v$ and $u \not\sim w$. By Payne and Thas [13, 1.3.6], the triad $\{x, y, v\}$ has either 1 or $s + 1$ centers. Now, the points u and w are collinear with y and v , but not with x . So, the triad $\{x, y, v\}$ has a unique center a_1 . In a similar way, one shows that the triad $\{x, u, w\}$ has a unique center a_2 . Obviously, $a_1 \neq a_2$. If the points x, a_1 and a_2 are collinear for all possible choices for u, v and w such that $y \sim u \sim v \sim w \sim y$, $x \not\sim u$, $x \not\sim v$, $x \not\sim w$, $y \not\sim v$ and $u \not\sim w$, then we say that the pair (Q, x) satisfies property (P_y) .

2 Generalized Admissible Triples

2.1 Definition

A *generalized admissible triple* is a triple $T = (\mathcal{L}, X, \Delta)$, where

- X is a set of size at least 2. We put $s := |X| - 1$. Let $Sym(X)$ denote the group of all permutations of X . If $\sigma_1, \sigma_2 \in Sym(X)$, then we denote $\sigma_2 \circ \sigma_1$ also by $\sigma_1\sigma_2$. We denote the trivial automorphism of X by 1.
- \mathcal{L} is a Steiner system $S(2, s + 1, st + 1)$ where $t \geq 1$. We denote the point set of \mathcal{L} by P .
- Δ is a map from $P \times P$ to $Sym(X)$ such that the following conditions hold for any three points p, q and r of \mathcal{L} :

(GAT1) If p, q and r are collinear, then $\Delta(p, q)\Delta(q, r) = \Delta(p, r)$.

(GAT2) If p, q and r are not collinear, then the permutation $\Delta(p, q)\Delta(q, r)\Delta(r, p)$ has no fixpoints.

Suppose that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple.

- By taking $p = q$ in (GAT1), we see that $\Delta(p, p) = 1$ for every point p of \mathcal{L} .
- By taking $p = r$ in (GAT1), we then see that $\Delta(q, p) = \Delta(p, q)^{-1}$ for every two points p and q of \mathcal{L} .

2.2 Coordinatization of generalized quadrangles with a regular spread

Proposition 2.1 ([6, Theorem 4]) *Suppose that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple and let P denote the point set of \mathcal{L} . Let Γ be the graph with vertex set $X \times P$, with two vertices (x_1, p_1) and (x_2, p_2) adjacent whenever either*

- (a) $p_1 = p_2$ and $x_1 \neq x_2$, or
- (b) $p_1 \neq p_2$ and $x_2 = x_1^{\Delta(p_1, p_2)}$.

Then Γ is the collinearity graph of a generalized quadrangle Q_T . Moreover, the set $L_p := \{(x, p) \mid x \in X\}$ is a line of Q_T for every point p of \mathcal{L} and the lines L_p , $p \in P$, form a regular spread S_T of Q_T .

Definitions. (1) For every generalized admissible triple T , we define $\Omega(T) := (Q_T, S_T)$ where Q_T and S_T are as in Proposition 2.1.

(2) Let Q_1 and Q_2 be two generalized quadrangles and let S_i , $i \in \{1, 2\}$, be a regular spread of Q_i . Then (Q_1, S_1) and (Q_2, S_2) are said to be *equivalent* if there exists an isomorphism from Q_1 to Q_2 mapping S_1 to S_2 .

Remark. Suppose $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple. Then there are two types of lines in Q_T . Lines belonging to S_T are of the form $L_p = \{(x, p) \mid x \in X\}$ for some point p of \mathcal{L} . The remaining lines are of the form $L[x, p, M] := \{(x^{\Delta(p, r)}, r) \mid r \in M\}$ for some incident point-line pair (p, M) of \mathcal{L} and some $x \in X$.

Proposition 2.2 ([6, Theorem 6]) *Let Q be a generalized quadrangle of order (s, t) with a regular spread S . Then there exists a generalized admissible triple T such that $\Omega(T)$ is equivalent with (Q, S) .*

3 Equivalence of Generalized Admissible Triples

Two generalized admissible triples $T = (\mathcal{L}, X, \Delta)$ and $T' = (\mathcal{L}', X', \Delta')$ are called *isomorphic* if there exists an isomorphism α from \mathcal{L} to \mathcal{L}' and a bijection β between X and X' such that $\Delta'(p^\alpha, q^\alpha) = \beta^{-1} \Delta(p, q) \beta$ for all points p and q of \mathcal{L} . If T and T' are isomorphic, then $\Omega(T)$ and $\Omega(T')$ are equivalent. The converse is not necessarily true:

Proposition 3.1 *Let $T = (\mathcal{L}, X, \Delta)$ be a generalized admissible triple. Let \mathcal{L}' be a linear space isomorphic to \mathcal{L} and let X' denote a set of the same size as X . Let α denote an isomorphism from \mathcal{L} to \mathcal{L}' and let θ_p be a bijection from X to X' for every point p of \mathcal{L} . For any two points p and q of \mathcal{L} , we define*

$$\Delta'(p^\alpha, q^\alpha) = \theta_p^{-1} \Delta(p, q) \theta_q.$$

Then $T' = (\mathcal{L}', X', \Delta')$ is a generalized admissible triple and $\Omega(T')$ is equivalent with $\Omega(T)$.

Proof. For all points p, q and r of \mathcal{L} , we have $\Delta'(p^\alpha, q^\alpha) \Delta'(q^\alpha, r^\alpha) \Delta'(r^\alpha, p^\alpha) = \theta_p^{-1} \Delta(p, q) \Delta(q, r) \Delta(r, p) \theta_p$. So, we have

- $\Delta'(p^\alpha, q^\alpha) \Delta'(q^\alpha, r^\alpha) \Delta'(r^\alpha, p^\alpha)$ is trivial if and only if $\Delta(p, q) \Delta(q, r) \Delta(r, p)$ is trivial.
- $\Delta'(p^\alpha, q^\alpha) \Delta'(q^\alpha, r^\alpha) \Delta'(r^\alpha, p^\alpha)$ has a fixpoint if and only if $\Delta(p, q) \Delta(q, r) \Delta(r, p)$ has a fixpoint.

It follows that T' is an admissible triple. Now, put $\Omega(T) = (Q_T, S_T)$ and $\Omega(T') = (Q_{T'}, S_{T'})$. For every point (x, p) of Q_T , we define $(x, p)^\beta := (x^{\theta_p}, p^\alpha)$. Then β is a bijection between the point sets of Q_T and $Q_{T'}$. Also, every line of S_T is mapped to a line of $S_{T'}$. We will now show that β maps collinear points (x_1, p_1) and (x_2, p_2) of Q_T to collinear points $(x_1, p_1)^\beta$ and $(x_2, p_2)^\beta$ of $Q_{T'}$. The proposition then follows from the fact that Q_T and $Q_{T'}$ have the same order. Obviously, $(x_1, p_1)^\beta$ and $(x_2, p_2)^\beta$ are collinear if $p_1 = p_2$. So, suppose that $p_1 \neq p_2$. Then $(x_1, p_1)^\beta = (x_1^{\theta_{p_1}}, p_1^\alpha)$ and $(x_2, p_2)^\beta = (x_1^{\Delta(p_1, p_2) \theta_{p_2}}, p_2^\alpha)$. Since $\theta_{p_1} \Delta'(p_1^\alpha, p_2^\alpha) = \Delta(p_1, p_2) \theta_{p_2}$, the points $(x_1, p_1)^\beta$ and $(x_2, p_2)^\beta$ are collinear. This proves the proposition. \square

Definition. We say that two generalized admissible triples $T = (\mathcal{L}, X, \Delta)$ and $T' = (\mathcal{L}', X', \Delta')$ are *equivalent* if there exist

- an isomorphism α from \mathcal{L} to \mathcal{L}' ,
- a bijection θ_p between X and X' for every point p of \mathcal{L} ,

such that

$$\Delta'(p^\alpha, q^\alpha) = \theta_p^{-1} \Delta(p, q) \theta_q$$

for all points p and q of \mathcal{L} . Isomorphism is a special type of equivalence.

Proposition 3.2 *Suppose $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple and that o is a point of \mathcal{L} . Then there exists a generalized admissible triple $T' = (\mathcal{L}, X, \Delta')$ equivalent with T such that $\Delta'(o, p) = 1$ for every point p of \mathcal{L} .*

Proof. In the previous definition, let α be the trivial automorphism of \mathcal{L} and put $\theta_p := \Delta(o, p)^{-1}$ for every point p of \mathcal{L} . Then $\Delta'(o, p) = \Delta(o, o) \Delta(o, p) \Delta(o, p)^{-1} = 1$ for every point p of \mathcal{L} . \square

Proposition 3.3 ([6, Theorem 8]) *Let T and T' denote two generalized admissible triples. Then $\Omega(T)$ is equivalent with $\Omega(T')$ if and only if T and T' are equivalent.*

4 Admissible triples

A generalized admissible triple $T = (\mathcal{L}, X, \Delta)$ is called an *admissible triple* if there exists a binary operation \cdot on the set X such that

- (AT1) (X, \cdot) is a (multiplicative) group,
- (AT2) $x^{-1} \cdot x^{\Delta(p_1, p_2)}$ only depends on the points p_1 and p_2 of \mathcal{L} and not on the element x of X .

Suppose now that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple. Let P be the point set of \mathcal{L} and put $s := |X| - 1$. For every point p of \mathcal{L} , we define $\Omega(p) := \{\Delta(p, q) \mid q \in P\}$. Then $\langle \text{Im}(\Delta) \rangle := \langle \Delta(x, y) \mid x, y \in P \rangle$ is equal to $\langle \bigcup_{p \in P} \Omega(p) \rangle$. The following proposition was proved in De Bruyn [6] (Theorems 10, 11, 12 and Corollary 1).

Proposition 4.1 ([6]) (1) *If \mathcal{L} is not a line, then $\langle \text{Im}(\Delta) \rangle$ acts transitively on X . So, $|\langle \text{Im}(\Delta) \rangle| \geq s + 1$.*

(2) *Suppose that \mathcal{L} is not a line and that T is an admissible triple. Then $|\langle \text{Im}(\Delta) \rangle| = s + 1$.*

(3) *Suppose \mathcal{L} is not a line and $|\langle \text{Im}(\Delta) \rangle| = s + 1$. Then $\text{Im}(\Delta)$ acts regularly on X and T is an admissible triple.*

Proposition 4.2 ([6, Theorem 13]) *Suppose that \mathcal{L} is not a line and that o is a point of \mathcal{L} such that $\Delta(o, p) = 1$ for every point p of \mathcal{L} . Then $|\Omega(p)| \geq s + 1$ for every point $p \neq o$ of \mathcal{L} . Moreover, if T is an admissible triple, then $\Omega(p) = \langle \text{Im}(\Delta) \rangle$ for every point $p \neq o$ of \mathcal{L} .*

Proposition 4.3 ([6, Theorem 14]) *Suppose (i) \mathcal{L} is not a line, (ii) there exists a point o such that $\Delta(o, p) = 1$ for every point p of \mathcal{L} , (iii) there exists a subset Ω of size $s + 1$ of $\text{Sym}(X)$ such that $\Omega(p) = \Omega$ for every point $p \neq o$ of \mathcal{L} . Then $\langle \text{Im}(\Delta) \rangle = \Omega$ and T is an admissible triple.*

5 Application of generalized admissible triples

W. M. Kantor [10] and K. Thas [14] independently proved the following result.

Proposition 5.1 ([10], [14]) *Let Q be a generalized quadrangle of order $s \geq 2$. If Q has a hyperbolic line all of whose points are centers of symmetry, then s is a prime power and Q is isomorphic to $W(s)$.*

The theory of generalized admissible triples can be used to improve that result as follows:

Proposition 5.2 ([6]) *Let Q be a generalized quadrangle of order $s \geq 2$. If Q has a hyperbolic line all of whose points are regular, then all these points are also centers of symmetry. Hence, s is a prime power and Q is isomorphic to $W(s)$.*

Proposition 5.2 is a characterization of the symplectic generalized quadrangle $W(s)$ in terms of a nice set of $s + 1$ regular points. Previous characterization results of this type all needed roughly s^2 points:

Proposition 5.3 ([13, 1.3.6, 5.2.5 and 5.2.6]) *Let Q be a generalized quadrangle of order $s \geq 2$.*

- (1) *If there exists a point x such that every point of $x^\perp \setminus \{x\}$ is regular, then s is a prime power and Q is isomorphic to $W(s)$.*
- (2) *If there exists an ovoid O , each point of which is regular, then s is an even prime power and Q is isomorphic to $W(s)$.*
- (3) *If there exists a subquadrangle of order $(s, 1)$, each point of which is regular, then s is an even prime power and Q is isomorphic to $W(s)$.*

Proposition 5.2 was proved in the paper [6]. We give here a sketch of the proof of [6].

Suppose Q is a generalized quadrangle of order $s \geq 2$ and that H is a hyperbolic line of Q all of whose points are regular. Let x be an arbitrary point of H . If we expand Q about the point x , then we obtain a generalized quadrangle $P(Q, x)$ of order $(s - 1, s + 1)$. The hyperbolic lines of Q through x define a regular spread $S(Q, x)$ of $P(Q, x)$. By Proposition 2.2, there exists

a generalized admissible triple $T = (\mathcal{L}, X, \Delta)$ such that $\Omega(T) = (Q_T, S_T)$ is equivalent with $(P(Q, x), S(Q, x))$. The Steiner system \mathcal{L} is isomorphic to the incidence structure whose points are the elements of $S(Q, x)$ and whose lines are the sets $\{K, L\}^{\perp\perp}$ (calculated in the generalized quadrangle $P(Q, x)$) where K and L are two distinct elements of $S(Q, x)$. Since $P(Q, x)$ has order $(s-1, s+1)$, \mathcal{L} is a Steiner system of type $S(2, s, s^2)$, i.e. an affine plane of order s . We denote the point of \mathcal{L} corresponding with H by o . By Proposition 3.2, we may without loss of generality suppose that

(I) $\Delta(o, p) = 1$ for all points p of \mathcal{L} .

The fact that T is a generalized admissible triple gives rise to the following conditions:

(II) If p, q and r are three collinear points of \mathcal{L} , then $\Delta(p, q) \Delta(q, r) = \Delta(p, r)$.

(III) If p, q and r are points of \mathcal{L} such that the permutation $\Delta(p, q) \Delta(q, r) \Delta(r, p)$ has at least one fixpoint, then p, q and r are collinear.

The following proposition was shown in De Bruyn and Payne [7, Theorem 2.1].

Proposition 5.4 ([7]) *Let Q be a generalized quadrangle of order $s \geq 2$ with a regular point x and let y be a point of Q noncollinear with x . Then the pair (Q, x) satisfies property (P_y) if and only if y is a regular point.*

Using Proposition 5.4, it is possible to show that also the following condition must be satisfied:

(IV) if o, p, q and r are mutually distinct points of \mathcal{L} such that $\Delta(p, r) \Delta(r, q)$ has at least one fixpoint, then the lines or and pq are parallel.

Using conditions (I), (II), (III), (IV) and Proposition 4.3, one can show (see [6]) that the generalized admissible triple T is an admissible triple. By Proposition 6.1 which we will discuss in Section 6, this implies that $S(Q, x)$ is a spread of symmetry of $P(Q, x)$. By De Soete and Thas [9, Theorem 2.7], this implies that x is a center of symmetry of Q .

Now, x was an arbitrary point of H . So, all points of H must be centers of symmetry. By the main results of Kantor [10] and K. Thas [14], we then have that Q is isomorphic to $W(s)$.

6 Equivalent definition of admissible triple

Suppose $T = (\mathcal{L}, X, \Delta)$ is an admissible triple, where \mathcal{L} is not a line. We put $s + 1 = |X|$. By Proposition 4.1, the subgroup $G := \langle \text{Im}(\Delta) \rangle$ of $\text{Sym}(X)$ has order $s + 1$. Putting $x_{p_1, p_2} := x^{-1} \cdot x^{\Delta(p_1, p_2)}$ (for an arbitrary $x \in X$), we see that $x^{\Delta(p_1, p_2)} = x \cdot x_{p_1, p_2}$ for every $x \in X$ and all points p_1, p_2 of \mathcal{L} . Conditions (GAT1) and (GAT2) then become: three points p, q and r of \mathcal{L} are collinear if and only if $x_{pq} \cdot x_{qr} = x_{pr}$. This leads us to a definition of the notion *admissible triple* which is slightly different, but completely equivalent to the one given before.

Definition. An *admissible triple* is a triple $T = (\mathcal{L}, G, \Delta)$, where:

- G is a nontrivial group of order $s + 1 \geq 2$.
- \mathcal{L} is a Steiner system $S(2, s + 1, st + 1)$ where $t \geq 1$. We denote the point set of \mathcal{L} by P .
- Δ is a map from $P \times P$ to G such that the following holds for any three points x, y and z of \mathcal{L} :

$$(\text{AT}) \quad x, y \text{ and } z \text{ are collinear} \Leftrightarrow \Delta(x, y) + \Delta(y, z) = \Delta(x, z).$$

If T is an admissible triple, then $\Delta(x, x) = 0$ and $\Delta(y, x) = -\Delta(x, y)$ for all points x and y of \mathcal{L} . Notice that we have used the additive notation for the group G . The following proposition was shown in De Bruyn [1, Theorems 3.1 and 3.2].

Proposition 6.1 ([1]) *Suppose that $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and let P denote the point set of \mathcal{L} . Let Γ be the graph with vertex set $G \times P$, two vertices (g_1, x_1) and (g_2, x_2) being adjacent whenever either*

- $x_1 = x_2$ and $g_1 \neq g_2$, or
- $x_1 \neq x_2$ and $g_2 = g_1 + \Delta(x_1, x_2)$.

Then Γ is the collinearity graph of a generalized quadrangle Q_T . Moreover, the set $L_x := \{(g, x) \mid g \in G\}$ is a line of Q_T for every point x of \mathcal{L} and the lines L_x , $x \in P$, form a spread of symmetry S_T of Q_T .

Example. Let \mathcal{L} be the Desarguesian affine plane $\text{AG}(2, q)$ coordinatized in the natural way by the finite field \mathbb{F}_q . Let G be the additive group of \mathbb{F}_q . For all points (x_1, y_1) and (x_2, y_2) of \mathcal{L} , we define $\Delta((x_1, y_1), (x_2, y_2)) := \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. Now, three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) of \mathcal{L} are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

or equivalently, if and only if $\Delta((x_1, y_1), (x_3, y_3)) = \Delta((x_1, y_1), (x_2, y_2)) + \Delta((x_2, y_2), (x_3, y_3))$. We will meet this example later again in Section 8.3.

Definition. For every admissible triple T , we define $\Omega(T) := (Q_T, S_T)$ where Q_T and S_T are as in Proposition 6.1. The following proposition has been shown in De Bruyn [1, Theorem 4.2].

Proposition 6.2 ([1]) *If S is a spread of symmetry of a generalized quadrangle Q of order (s, t) , then there exists an admissible triple T such that $\Omega(T)$ is equivalent with (Q, S) .*

Remark. If S is a spread of symmetry of a generalized quadrangle Q of order (s, t) and if T is a generalized admissible triple such that $\Omega(T) = (Q_T, S_T)$ is equivalent with (Q, S) , then T is not necessarily an admissible triple.

Proposition 6.3 ([1, p761]) *Suppose that $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and let P denote the point set of \mathcal{L} . Let \mathcal{L}' be a linear space isomorphic to \mathcal{L} and let G' denote a group isomorphic to G . Let α denote an isomorphism from \mathcal{L} to \mathcal{L}' , let θ denote an isomorphism from G to G' and let f denote an arbitrary map from P to G . For all points x and y of \mathcal{L} , we define*

$$\Delta'(\alpha(x), \alpha(y)) := [f(x) + \Delta(x, y) - f(y)]^\theta.$$

Then $T' := (\mathcal{L}', G', \Delta')$ is an admissible triple and $\Omega(T')$ is equivalent with $\Omega(T)$.

Definition. Let $T_1 = (\mathcal{L}_1, G_1, \Delta_1)$ and $T_2 = (\mathcal{L}_2, G_2, \Delta_2)$ be two admissible triples. Let P_i , $i \in \{1, 2\}$, denote the point set of \mathcal{L}_i .

- If \mathcal{L}_1 and \mathcal{L}_2 are lines, then we say that T_1 and T_2 are equivalent if \mathcal{L}_1 and \mathcal{L}_2 contain the same number of points.
- If \mathcal{L}_1 or \mathcal{L}_2 is not a line, then we say that T_1 and T_2 are equivalent if
 - (i) there exists an isomorphism α from \mathcal{L}_1 to \mathcal{L}_2 ,
 - (ii) there exists an isomorphism θ from G_1 to G_2 ,
 - (iii) there exists a map $f : P_1 \rightarrow G_1$

such that

$$\Delta_2(\alpha(x), \alpha(y)) = [f(x) + \Delta_1(x, y) - f(y)]^\theta$$

for all points x and y of \mathcal{L}_1 .

Proposition 6.4 ([8, Theorem 2.1]) *Two admissible triples T_1 and T_2 are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent.*

Similarly, as in Proposition 3.2, one can show that:

Proposition 6.5 *If $T = (\mathcal{L}, G, \Delta)$ is an admissible triple and if o is an arbitrary point of \mathcal{L} , then there exists an admissible triple $T' = (\mathcal{L}, G, \Delta')$ equivalent with T such that $\Delta'(o, p) = 0$ for every point p of \mathcal{L} .*

7 Applications of admissible triples

7.1 Generalized quadrangles of order 5 with a center of symmetry

The theory of admissible triples can be used to classify all generalized quadrangles of order 5 with a center of symmetry. The key to do that is the following result proved in De Bruyn [5].

Proposition 7.1 ([5]) *Up to equivalence there exists a unique admissible triple $T = (\mathcal{L}, G, \Delta)$, where \mathcal{L} is an affine plane of order 5.*

If $T = (\mathcal{L}, G, \Delta)$ is an admissible triple coordinatizing a generalized quadrangle of order $(4, 6)$ with a spread of symmetry, then \mathcal{L} is a Steiner system $S(2, 5, 25)$ and hence an affine plane of order 5. So, we can say the following:

Corollary 7.2 *Up to equivalence, there exists a unique pair (Q, S) , where Q is a generalized quadrangle of order $(4, 6)$ and S is a spread of symmetry of Q .*

If S^* is a spread of symmetry of a generalized quadrangle Q^* of order $(s - 1, s + 1)$, then by De Soete and Thas [9, Theorems 2.7 and 2.8], there exists up to isomorphism a unique pair (Q, x) with Q a generalized quadrangle of order s and x a center of symmetry of Q , such that $(P(Q, x), S(Q, x))$ is equivalent with (Q^*, S^*) . Hence, we can say the following:

Corollary 7.3 *Every generalized quadrangle of order 5 with a center of symmetry is isomorphic to $W(5)$.*

7.2 Coordinatization of glued near hexagons

7.2.1 Definition of glued near hexagon

For every $i \in \{1, 2\}$, let Q_i be a generalized quadrangle of order (s, t_i) and let S_i be a spread of Q_i . We put

$$\begin{aligned} S_1 &= \{L_1^{(1)}, L_2^{(1)}, \dots, L_{1+st_1}^{(1)}\}, \\ S_2 &= \{L_1^{(2)}, L_2^{(2)}, \dots, L_{1+st_2}^{(2)}\}. \end{aligned}$$

The line $L_1^{(i)}$, $i \in \{1, 2\}$, is called the *base line* of the spread S_i . For every $i \in \{1, 2\}$ and all $j, k \in \{1, \dots, 1 + st_i\}$, let $\Phi_{j,k}^{(i)}$ denote the bijection from the line $L_j^{(i)}$ to the line $L_k^{(i)}$ which maps each point x of $L_j^{(i)}$ to the unique point of $L_k^{(i)}$ nearest to x .

For every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$, one can define a graph Γ_θ with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two distinct vertices $(x, L_i^{(1)}, L_j^{(2)})$ and $(y, L_k^{(1)}, L_l^{(2)})$ are adjacent whenever precisely one of the following three conditions is satisfied:

- (1) $i = k$, $j = l$ and $x \neq y$;
- (2) $i \neq k$, $j = l$ and $\Phi_{1,i}^{(1)}(x)$, $\Phi_{1,k}^{(1)}(y)$ are collinear points of Q_1 ;
- (3) $i = k$, $j \neq l$ and $\Phi_{1,j}^{(2)} \circ \theta(x)$, $\Phi_{1,l}^{(2)} \circ \theta(y)$ are collinear points of Q_2 .

Proposition 7.4 ([2, Lemma 3.1]) *Through every two adjacent vertices of Γ_θ , there exists a unique maximal clique.*

Definition. Let \mathcal{S}_θ be the incidence structure with points the vertices of Γ_θ , with lines the maximal cliques of Γ_θ and with containment as incidence relation.

Definitions: • For all $i, j \in \{1, \dots, 1 + st_1\}$, let $\phi_{i,j}^{(1)}$ be the permutation $\Phi_{j,1}^{(1)} \circ \Phi_{i,j}^{(1)} \circ \Phi_{1,i}^{(1)}$ of $L_1^{(1)}$. The group of permutations of $L_1^{(1)}$ generated by all elements $\phi_{i,j}^{(1)}$ is denoted by G_1 .

• For all $i, j \in \{1, \dots, 1 + st_2\}$, let $\phi_{i,j}^{(2)}$ be the permutation $\theta^{-1} \circ \Phi_{j,1}^{(2)} \circ \Phi_{i,j}^{(2)} \circ \Phi_{1,i}^{(2)} \circ \theta$ of $L_1^{(1)}$. The group of permutations of $L_1^{(1)}$ generated by all elements $\phi_{i,j}^{(2)}$ is denoted by G_2 .

Proposition 7.5 ([2, Theorem 3.7]) *\mathcal{S}_θ is a near hexagon if and only if every element of G_1 commutes with every element of G_2 .*

Proposition 7.6 ([2, p216]) *If \mathcal{S}_θ is a near hexagon, then there exist partitions T_1 and T_2 of \mathcal{S}_θ in subspaces such that:*

- (1) *the partial linear space induced by every element of T_1 is isomorphic to Q_1 ;*
- (2) *the partial linear space induced by every element of T_2 is isomorphic to Q_2 ;*
- (3) *every element of T_1 intersects every element of T_2 in a line;*
- (4) *every line of \mathcal{S}_θ is contained in an element of $T_1 \cup T_2$.*

Definition. If the condition of Proposition 7.5 is satisfied, then \mathcal{S}_θ is called a *glued near hexagon*.

Example. Suppose Q_2 is an $(s+1) \times (s+1)$ -grid and let S_2 be one of the two spreads of Q_2 . Then the condition of Proposition 7.5 is satisfied for every bijection θ between $L_1^{(1)}$ and $L_1^{(2)}$. The corresponding glued near hexagon \mathcal{S}_θ is isomorphic to $Q_1 \times \mathbb{L}_{s+1}$. We call \mathcal{S}_θ a *trivial glued near hexagon*.

Proposition 7.7 ([1, Section 9]) *If \mathcal{S}_θ is a nontrivial glued near hexagon, then S_1 and S_2 are spreads of symmetry of respectively Q_1 and Q_2 .*

Remark. From the construction above, one might get the impression that the choice of the base lines $L_1^{(1)}$ and $L_1^{(2)}$ in the respective spreads S_1 and S_2 plays an important role. This is however not the case. If \mathcal{S}_θ is a (glued) near hexagon, then it can also be obtained from any choice of the base lines in S_1 and S_2 (by changing the map θ accordingly).

7.2.2 Coordinatization

Suppose $T_1 = (\mathcal{L}_1, G_1, \Delta_1)$ and $T_2 = (\mathcal{L}_2, G_2, \Delta_2)$ are two admissible triples, with G_1 and G_2 isomorphic groups. Let P_i , $i \in \{1, 2\}$, denote the point-set of \mathcal{L}_i . For every anti-automorphism $\theta : G_1 \rightarrow G_2$, we can define a graph Γ_θ with vertex set $G_1 \times P_1 \times P_2$. Two vertices (g, p_1, p_2) and (g', p'_1, p'_2) are adjacent if and only if precisely one of the following conditions is satisfied (multiplicative notation for the groups):

- (1) $p_1 = p'_1$, $p_2 = p'_2$ and $g \neq g'$;
- (2) $p_1 = p'_1$, $p_2 \neq p'_2$ and $g' = \theta^{-1}(\Delta_2(p_2, p'_2)) \cdot g$;
- (3) $p_1 \neq p'_1$, $p_2 = p'_2$ and $g' = g \cdot \Delta_1(p_1, p'_1)$.

Proposition 7.8 ([4, Section 4.7.4]) (1) *Every two adjacent vertices of Γ_θ are contained in a unique maximal clique.*

(2) *The point-line incidence structure \mathcal{S}_θ with points the vertices of Γ_θ , with lines the maximal cliques of Γ_θ (natural incidence) is a glued near hexagon.*

(3) *Every nontrivial glued near hexagon can be obtained in the way described above.*

8 Known examples of generalized quadrangles with a spread of symmetry

In this section, we give a description of all known generalized quadrangles with a spread of symmetry. We also give the corresponding admissible triples (up to equivalence). For more details about the computations, see De Bruyn [3].

8.1 The grids

The $(s+1) \times (s+1)$ -grid $\mathbb{L}_{s+1} \times \mathbb{L}_{s+1}$ has two spreads S_1 and S_2 . These spreads are spreads of symmetry and are equivalent under the automorphism group of the grid. Hence, there exists an admissible triple T such that (Q_T, S_T) is equivalent with $(\mathbb{L}_{s+1} \times \mathbb{L}_{s+1}, S_1)$.

We now give a description of this admissible triple $T = (\mathcal{L}, G, \Delta)$ which is determined up to equivalence:

- \mathcal{L} is a line of size $s + 1 \geq 2$;
- G is an arbitrary group of order $s + 1$;
- for all points x and y of \mathcal{L} , $\Delta(x, y)$ is the neutral element 0 of G .

8.2 The dual grids

The point-line dual $K_{t+1, t+1}$ of the $(t+1) \times (t+1)$ -grid has $(t+1)!$ spreads. All these spreads are spreads of symmetry and equivalent under the automorphism group of $K_{t+1, t+1}$. Hence, there exists an admissible triple T such that (Q_T, S_T) is equivalent with $(K_{t+1, t+1}, S)$, where S is an arbitrary spread of $K_{t+1, t+1}$.

We now give a description of this admissible triple $T = (\mathcal{L}, G, \Delta)$ which is determined up to equivalence:

- \mathcal{L} is the complete graph on $t + 1$ vertices (regarded as linear space);
- G is the group of order 2;
- for any two points x and y of \mathcal{L} , $\Delta(x, y) = 0$ if and only if $x = y$.

8.3 The generalized quadrangle $P(W(q), x)$

As remarked above, every point x of the symplectic generalized quadrangle $W(q)$ is a regular point. So, if we expand $W(q)$ about the point x , then we obtain a generalized quadrangle $P(W(q), x)$ of order $(q - 1, q + 1)$. Since the automorphism group of $W(q)$ acts transitively on its point set, there is essentially one generalized quadrangle which arises in this way. The set $S(W(q), x)$ of hyperbolic lines through x defines a spread of symmetry of $P(W(q), x)$. Hence, there exists an admissible triple T such that $(P(W(q), x), S(W(q), x))$ is equivalent with $\Omega(T)$.

We now give a description of this admissible triple $T = (\mathcal{L}, G, \Delta)$ which is determined up to equivalence:

- \mathcal{L} is the Desarguesian affine plane $AG(2, q)$ with point set $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{F}_q\}$;
- G is the additive group of the finite field \mathbb{F}_q ;
- for any two points (x_1, y_1) and (x_2, y_2) of \mathcal{L} ,

$$\Delta((x_1, y_1), (x_2, y_2)) := \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

8.4 The generalized quadrangle $Q(5, q)$

Up to projective equivalence, there exists a unique quadric Q in $PG(5, q)$ whose maximal subspaces have dimension 1 (so-called Witt-index 2). The equation of such a quadric with respect to a suitable reference system is given by

$$f(X_0, X_1) + X_2X_3 + X_4X_5 = 0.$$

Here f is an irreducible quadratic polynomial in two variables. The points and lines contained in such an elliptic quadric define a generalized quadrangle which we will denote by $Q(5, q)$.

Up to projective equivalence, there exists a unique Hermitian variety H in $PG(3, q^2)$ whose maximal subspaces have dimension 1 (so-called Witt-index 2). The equation of such a Hermitian variety with respect to a suitable reference system is given by

$$X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0.$$

The points and lines contained in such a Hermitian variety define a generalized quadrangle which we will denote by $H(3, q^2)$. The point-line dual of $H(3, q^2)$ is isomorphic to $Q(5, q)$. If we consider a hyperplane π in $PG(3, q^2)$ which is nontangent to H , then $H \cap \pi$ is a so-called *unital* of π , i.e. a non-singular Hermitian variety (of Witt-index 1) in a plane. It is easily seen that $H \cap \pi$ is an ovoid of $H(3, q^2)$. One can show that this ovoid is an ovoid of symmetry. Hence, it dualizes to a spread of symmetry S of $Q(5, q)$. It follows that there exists an admissible triple $T = (\mathcal{L}, G, \Delta)$ such that $\Omega(T)$ is equivalent with $(Q(5, q), S)$. We now give a description of this admissible triple $T = (\mathcal{L}, G, \Delta)$ which is determined up to equivalence.

Let V be a 3-dimensional vector space over \mathbb{F}_{q^2} equipped with a nonsingular Hermitian form (\cdot, \cdot) which is linear in the first argument and semi-linear in the second. The totally isotropic 1-spaces of V define in unital U in $\text{PG}(V) \cong \text{PG}(2, q^2)$. The lines of $\text{PG}(V)$ which contain precisely $q+1$ points of U define a Steiner system $\mathcal{L} = S(2, q+1, q^3+1)$ on the point-set U . Let $G := \{x \in \mathbb{F}_{q^2} \mid x^{q+1} = 1\}$ with multiplication inherited from \mathbb{F}_{q^2} . Let $z = \langle \bar{a} \rangle$ be a given point of U . For two points $x = \langle \bar{b} \rangle$ and $y = \langle \bar{c} \rangle$ of U , we define

$$\Delta(x, y) = \begin{cases} -(\bar{a}, \bar{b})^{q-1}(\bar{b}, \bar{c})^{q-1}(\bar{c}, \bar{a})^{q-1} \in G & \text{if } x, y, z \text{ are mutually different,} \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $\Delta(x, y)$ is well-defined. If we replace \bar{b} by $\mu\bar{b}$ and \bar{c} by $\lambda\bar{c}$ with $\mu, \lambda \in \mathbb{F}_{q^2} \setminus \{0\}$, then the value of $\Delta(x, y)$ is unaltered.

8.5 The generalized quadrangle $T_2^*(H)$

A *hyperoval* in a projective plane of order q is a set of $q+2$ points which intersects each line in either 0 or 2 points. A necessary condition for the existence of a hyperoval is that q is even. Hyperovals in the Desarguesian projective plane $\text{PG}(2, q)$, q even, do exist: take a conic union its nucleus (which is the intersection of all tangent lines).

Suppose H is a hyperoval of a Desarguesian projective plane $\text{PG}(2, q)$, q even. Embed $\text{PG}(2, q)$ as a hyperplane in the projective 3-space $\text{PG}(3, q)$. Then the following incidence structure $T_2^*(H)$ can be defined:

- The points of $T_2^*(H)$ are the affine points of $\text{PG}(3, q)$, i.e. the points of $\text{PG}(3, q) \setminus \text{PG}(2, q)$.
- The lines of $T_2^*(H)$ are the lines of $\text{PG}(3, q)$ which are not contained in $\text{PG}(2, q)$ and which intersect $\text{PG}(2, q)$ in a point of H .
- Incidence is derived from $\text{PG}(3, q)$.

The incidence structure $T_2^*(H)$ is a generalized quadrangle of order $(q-1, q+1)$. For every point x of H , the set S_x of lines of $\text{PG}(3, q)$ through x not contained in $\text{PG}(2, q)$ defines a spread S_x of $T_2^*(H)$. This spread S_x is a spread of symmetry. Hence, there exists an admissible triple $T = (\mathcal{L}, G, \Delta)$ such that (Q_T, S_T) is equivalent with $(T_2^*(H), S_x)$. We now give a description of T which is determined up to equivalence.

We will coordinatize the projective plane $\text{PG}(2, q)$ in such a way that

- $x = (1, 0, 0)$,
- $H = \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) \mid \lambda \in \mathbb{F}_q\}$,

Since H is a hyperoval, the function $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ must be bijective and satisfy the following property:

$$\begin{vmatrix} f(\lambda_1) & \lambda_1 & 1 \\ f(\lambda_2) & \lambda_2 & 1 \\ f(\lambda_3) & \lambda_3 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

Now, let \mathcal{L} be the Desarguesian affine plane $\text{AG}(2, q)$ with point set $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{F}_q\}$ and let G be the additive group of the finite field \mathbb{F}_q . We define Δ as follows: if (x_1, y_1) and (x_2, y_2) are two points of $\text{AG}(2, q)$, then

$$\Delta((x_1, y_1), (x_2, y_2)) := \begin{cases} (x_1 - x_2) \cdot f\left(\frac{y_1 - y_2}{x_1 - x_2}\right) & \text{if } x_1 \neq x_2, \\ 0 & \text{otherwise.} \end{cases}$$

8.6 The generalized quadrangle $(S_{xy}^-)^D$

The generalized quadrangle $(S_{xy}^-)^D$ is the point-line dual of the generalized quadrangle S_{xy}^- which first appeared in Payne [11]. We take the description of Payne [12]. Let H be a hyperoval of the projective plane $\pi = \text{PG}(2, q)$, q even. Again embed π as a hyperplane in $\text{PG}(3, q)$. The following generalized quadrangle S_{xy}^- of order $(q + 1, q - 1)$ can then be constructed. The points of S_{xy}^- are of three types:

- (i) points of $\text{PG}(3, q)$ not contained in π ;
- (ii) planes through x not containing y ;
- (iii) planes through y not containing x .

The lines of S_{xy}^- are those lines of $\text{PG}(3, q)$ which are not contained in π and which intersect $H \setminus \{x, y\}$ nontrivially. A point of S_{xy}^- and a line of S_{xy}^- are incident if and only if they are incident as objects of $\text{PG}(3, q)$. The set of points of type (ii) defines an ovoid of symmetry of S_{xy}^- and hence a spread of symmetry S of $(S_{xy}^-)^D$. Hence, there exists an admissible triple $T = (\mathcal{L}, G, \Delta)$ such that $\Omega(T)$ is equivalent with $((S_{xy}^-)^D, S)$.

We now give a description (up to equivalence) of this admissible triple T . We choose a reference system in $\text{PG}(2, q)$ such that $x = (1, 0, 0)$, $y = (0, 1, 0)$

and $H = \{(1, 0, 0), (0, 1, 0)\} \cup \{(f(\lambda), \lambda, 1) \mid \lambda \in \mathbb{F}_q\}$ for some permutation $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$ satisfying

$$\begin{vmatrix} f(\lambda_1) & \lambda_1 & 1 \\ f(\lambda_2) & \lambda_2 & 1 \\ f(\lambda_3) & \lambda_3 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

The Steiner system \mathcal{L} is the Desarguesian affine plane $\text{AG}(2, q)$ of order q with point set $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{F}_q\}$ and G is the additive group of the finite field \mathbb{F}_q . For any two points (x_1, y_1) and (x_2, y_2) of \mathcal{L} , we define

$$\Delta((x_1, y_1), (x_2, y_2)) := \begin{cases} (f(x_1) - f(x_2)) \frac{y_2 - y_1}{x_2 - x_1} & \text{if } x_1 \neq x_2, \\ 0 & \text{otherwise.} \end{cases}$$

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